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TORSION OF ELASTIC BODIES BOUNDED BY COORDINATE SURFACES OF TOROIDAL AND SPHERICAL COORDINATE SYSTEMS*

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On the basis of relationships between particular solutions of the torsion equation in toroidal and spherical coordinates, the Fourier method is applied to the solution of torsion contact problems for elastic bodies bounded by coordinate surfaces of toroidal and spherical coordinate systems.

1. Let $\alpha, \beta, \varphi; \alpha, \sigma, \varphi; r, \theta, \varphi; \rho, z, \varphi$ be toroidal, spherical, and cylindrical coordinates defined by the formulas /1–3/

$$x = ah_\beta^{-2}\operatorname{sh}\alpha\cos\varphi,\quad y = ah_\beta^{-2}\operatorname{sh}\alpha\sin\varphi,\quad z = ah_\beta^{-2}\sin\beta$$
$$x = ah_\sigma^{-2}\operatorname{sh}\alpha\cos\varphi,\quad y = ah_\sigma^{-2}\operatorname{sh}\alpha\sin\varphi,\quad z = ah_\sigma^{-2}\sin\sigma$$
$$x = r\sin\theta\cos\varphi,\quad y = r\sin\theta\sin\varphi,\quad z = r\cos\theta$$
$$x = \rho\cos\varphi,\ y = \rho\sin\varphi,\ z = z\ (a > 0,\ 0 \leqslant \alpha,\ \rho,\ r < \infty,$$
$$-\infty < z < \infty, -\pi < \beta,\ \sigma \leqslant \pi,\ 0 \leqslant \varphi < 2\pi,\ 0 \leqslant \theta \leqslant \pi,$$
$$h_\beta = \sqrt{\operatorname{ch}\alpha + \cos\beta},\quad h_\sigma = \sqrt{\operatorname{ch}\alpha - \cos\sigma})$$

The single displacement component different from zero $u = u_\varphi$ in problems of the pure torsion of elastic bodies of revolution satisfies the Eq.(1, 2, 4)

$$\frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} - \frac{u}{\rho^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.1}$$

Relationships between the particular solutions of (1.1) in spherical and toroidal coordinates results directly from the following equalities connecting the particular solutions of the Laplace equation in these coordinates (the factor $\cos m\varphi$ or $\sin m\varphi$ is omitted on both sides of each equality)

$$\left(\frac{a}{r}\right)^{2k+m+1} P_{2k+m}^m(\cos\theta) = h_\sigma \int_0^\infty C_{2k+m}^{(m)}(\tau)\, P_{-1/2+i\tau}^m(\operatorname{ch}\alpha)\operatorname{ch}\tau\sigma\, d\tau \tag{1.2}$$

$$\left(\frac{a}{r}\right)^{2k+m+2} P_{2k+m+1}^m(\cos\theta) = h_\sigma \int_0^\infty C_{2k+m+1}^{(m)}(\tau)\, P_{-1/2+i\tau}^m(\operatorname{ch}\alpha)\operatorname{sh}\tau\sigma\, d\tau$$

$$\left(\frac{r}{a}\right)^{2k+m} P_{2k+m}^m(\cos\theta) = h_\beta \int_0^\infty C_{2k+m}^{(m)}(\tau)\, P_{-1/2+i\tau}^m(\operatorname{ch}\alpha)\operatorname{ch}\tau\beta\, d\tau$$

$$\left(\frac{r}{a}\right)^{2k+m+1} P_{2k+m+1}^m(\cos\theta) = h_\beta \int_0^\infty C_{2k+m+1}^{(m)}(\tau)\, P_{-1/2+i\tau}^m(\operatorname{ch}\alpha)\operatorname{sh}\tau\beta\, d\tau$$
$$(|\beta| < \pi,\ |\sigma| < \pi)$$

$$h_\beta P_{-1/2+i\tau}^m(\operatorname{ch}\alpha)\operatorname{ch}\tau\beta = \sum_{n=0}^\infty D_{2n+m}^{(m)}(\tau)\left(\frac{r}{a}\right)^{2n+m} P_{2n+m}^m(\cos\theta)$$

$$h_\beta P_{-1/2+i\tau}^m(\operatorname{ch}\alpha)\operatorname{sh}\tau\beta = \sum_{n=0}^\infty D_{2n+m+1}^{(m)}(\tau)\left(\frac{r}{a}\right)^{2n+m+1} P_{2n+m+1}^m(\cos\theta)$$
$$(0 \leqslant r < a)$$

$$h_\sigma P_{-1/2+i\tau}^m(\operatorname{ch}\alpha)\operatorname{ch}\tau\sigma = \sum_{n=0}^\infty D_{2n+m}^{(m)}(\tau)\left(\frac{a}{r}\right)^{2n+m+1} P_{2n+m}^m(\cos\theta)$$

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$$h_\sigma P^m_{-1/2+i\tau}(\operatorname{ch}\alpha)\operatorname{sh}\tau\sigma = \sum_{n=0}^{\infty} D^{(m)}_{2n+m+1}(\tau)\left(\frac{a}{r}\right)^{2n+m+2} P^m_{2n+m+1}(\cos\theta)$$
$$(r > a)$$
$$\left(\frac{a}{r}\right)^{2k+m+1} P^m_{2k+m}(\cos\theta) = h_\sigma \sum_{n=-\infty}^{\infty} a^{(m)}_{2k+m,\,n} Q^m_{n-1/2}(\operatorname{ch}\alpha)\cos n\sigma$$
$$\left(\frac{a}{r}\right)^{2k+m+2} P^m_{2k+m+1}(\cos\theta) = h_\sigma \sum_{n=-\infty}^{\infty} a^{(m)}_{2k+m+1,\,n} Q^m_{n-1/2}(\operatorname{ch}\alpha)\, i\sin n\sigma$$
$$\left(\frac{r}{a}\right)^{2k+m} P^m_{2k+m}(\cos\theta) = h_\sigma \sum_{n=-\infty}^{\infty} (-1)^n a^{(m)}_{2k+m,\,n} Q^m_{n-1/2}(\operatorname{ch}\alpha)\cos n\sigma$$
$$\left(\frac{r}{a}\right)^{2k+m+1} P^m_{2k+m+1}(\cos\theta) = h_\sigma \sum_{n=-\infty}^{\infty} (-1)^{n+1} a^{(m)}_{2k+m+1,\,n} Q^m_{n-1/2} \times$$
$$(\operatorname{ch}\alpha)\, i\sin n\sigma$$
$$(\alpha > 0)$$
$$h_\sigma P^m_{n-1/2}(\operatorname{ch}\alpha)\cos n\sigma = \sum_{k=0}^{\infty} b^{(m)}_{2k+m,\,n}\left(\frac{a}{r}\right)^{2k+m+1} P^m_{2k+m}(\cos\theta)$$
$$h_\sigma P^m_{n-1/2}(\operatorname{ch}\alpha)\, i\sin n\sigma = \sum_{k=0}^{\infty} b^{(m)}_{2k+m+1,\,n}\left(\frac{a}{r}\right)^{2k+m+2} P^m_{2k+m+1}(\cos\theta)$$
$$(r > a)$$
$$h_\sigma P^m_{n-1/2}(\operatorname{ch}\alpha)\cos n\sigma = (-1)^n \sum_{k=0}^{\infty} b^{(m)}_{2k+m,\,n}\left(\frac{r}{a}\right)^{2k+m} P^m_{2k+m}(\cos\theta)$$
$$h_\sigma P^m_{n-1/2}(\operatorname{ch}\alpha)\, i\sin n\sigma =$$
$$(-1)^{n+1} \sum_{k=0}^{\infty} b^{(m)}_{2k+m+1,\,n}\left(\frac{r}{a}\right)^{2k+m+1} P^m_{2k+m+1}(\cos\theta)$$
$$(0 \leqslant r < a)$$

Here

$$C^{(m)}_n(\tau) = i^{n-m}\frac{2^{m+1/2}(n+m)!}{(2m)!\,(n-m)!}\frac{1}{\operatorname{ch}\pi\tau} F(1/2 + i\tau + m, m -$$
$$n; 2m+1; 2)$$
$$D^{(m)}_k(\tau) = \frac{(-1)^m i^{k-m} 2^{m+1/2}}{(2m)!}\frac{\Gamma(1/2 + i\tau + m)}{\Gamma(1/2 + i\tau - m)} F(1/2 + i\tau + m, m - k;$$
$$2m+1; 2)$$
$$a^{(m)}_{s,\,n} = \frac{1}{\pi} i^{s-m}(-1)^n \frac{2^{m+1/2}(s+m)!}{(2m)!\,(s-m)!} F(1/2 - n + m, m - s;$$
$$2m+1; 2)$$
$$b^{(m)}_{s,\,n} = \frac{(-1)^m i^{s-m} 2^{m+1/2}}{(2m)!}\frac{\Gamma(1/2 - n + m)}{\Gamma(1/2 - n - m)} F(1/2 - n + m, m - s;$$
$$2m+1; 2)$$
$$F(a, -n; c; z) = \sum_{m=0}^{n} \frac{(a)_m(-n)_m}{(c)_m\, m!} z^m$$
$$(\alpha)_m = \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)} = \alpha(\alpha+1)\ldots(\alpha+m-1)$$

$P_n^m(x)$ are the associated Legendre polynomials, $P_\nu^m(z)$, $Q_\nu^m(z)$ are associated Legendre functions of the first and second kind, $m - n \leqslant 0$, $m - k \leqslant 0$; $k, m, s = 0, 1, 2, \ldots$; $F(a, -n; c; z)$ is the hypergeometric polynomial in z, and $(\alpha)_m$ is the Pochhammer symbol /3, 5/.

The functions $\Gamma(1/2 + i\tau + m)/\Gamma(1/2 + i\tau - m)$, $C_n^m(\tau)$, $D_n^{(m)}(\tau)$ are real for real values of τ where

$$\frac{\Gamma(1/2 + i\tau + m)}{\Gamma(1/2 + i\tau - m)} = \frac{\Gamma(1/2 - i\tau + m)}{\Gamma(1/2 - i\tau - m)},\ C^{(m)}_n(-\tau) =$$
$$(-1)^{n-m} C^{(m)}_n(\tau),\ D^{(m)}_n(-\tau) = (-1)^{n-m} D^{(m)}_n(\tau)$$

The equalities (1.2) are obtained by solving special boundary value problems for the Laplace equation by the method described in /6/. For $m = 1$ they enable us to investigate a number of torsion problems of a) a sphere $0 \leqslant r \leqslant R$ with a cavity $\beta_1 \leqslant \beta \leqslant \beta_2$ $(\beta_1 < 0,\ \beta_2 > 0)$; b) a body $\beta_1 \leqslant \beta \leqslant \beta_2$ $(\beta_1 < 0,\ \beta_2 > 0)$ with a spherical cavity $0 \leqslant r \leqslant R$; c) a sphere $0 \leqslant r \leqslant R$ with a toroidal cavity $\alpha \geqslant \alpha_0 > 0$; d) a space with two cavities, one of which is bounded by a sphere $r = R$ and the other by the surface of a torus $\alpha = \alpha_0$, and certain other bodies bounded by coordinate surfaces of toroidal and spherical coordinate systems.

2. We will examine the problem of the equilibrium of 1) a truncated sphere $0 \leqslant \beta \leqslant \beta_0$ $(0 < \beta_0 \leqslant \pi)$ with a hemispherical depression $0 \leqslant r \leqslant R$, $0 \leqslant \theta \leqslant \pi/2$ clamped along the surface $\beta = \beta_0$ and twisted by a rigid stamp coupled to the surface $r = R$, $0 \leqslant \theta \leqslant \pi/2$; 2) a hemisphere $0 \leqslant r \leqslant R$, $0 \leqslant \theta \leqslant \pi/2$ with a segmental depression $0 \leqslant \beta \leqslant \beta_0$ $(0 < \beta_0 < \pi)$ clamped along the surface $r = R$, $0 \leqslant \theta \leqslant \pi/2$ and twisted by a rigid stamp coupled to the surface $\beta = \beta_0$; 3) a hemisphere $0 \leqslant r \leqslant R$, $0 \leqslant \theta \leqslant \pi/2$ with the toroidal depression $0 < \alpha_0 \leqslant \alpha < \infty$, $0 \leqslant \sigma \leqslant \pi$ clamped along the surface $r = R, 0 \leqslant \theta \leqslant \pi/2$ and twisted by a rigid stamp coupled to the surface of a torus along the section $\alpha = \alpha_0$, $0 \leqslant \sigma \leqslant \pi$.

The following boundary conditions correspond to these problems

$$1)\ u|_{\beta=\beta_0} = 0, \quad u|_{r=R} = \varepsilon\rho, \quad \frac{\partial u}{\partial z}\Big|_{z=0} = 0 \quad (R < \rho < a)$$

$$2)\ u|_{\sigma=\sigma_0} = \varepsilon\rho, \quad u|_{r=R} = 0, \quad \frac{\partial u}{\partial z}\Big|_{z=0} = 0 \quad (a < \rho < R)$$

$$3)\ u|_{\alpha=\alpha_0} = \varepsilon\rho, \quad u|_{r=R} = 0, \quad \frac{\partial u}{\partial z}\Big|_{z=0} = 0$$

$$\left(0 \leqslant \rho < a \operatorname{th}\frac{\alpha_0}{2}, \quad a \operatorname{cth}\frac{\alpha_0}{2} < \rho < R\right)$$

(ε is the angle of stamp rotation).

We represent the general solution of problem 1) as the sum of two components

$$u = h_\beta \int_0^\infty A(\tau) \operatorname{ch} \tau\beta P^1_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau + \sum_{n=0}^\infty B_n \left(\frac{R}{r}\right)^{2n+2} P^1_{2n+1}(\cos\theta)$$

each of which identically satisfies the condition

$$(\partial u / \partial z)|_{z=0} = 0 \ (R < \rho < a)$$

Using the equalities

$$\sigma = \pi - \beta \ (0 \leqslant \beta \leqslant \pi), \quad \rho = -rP_1{}^1(\cos\theta)$$

$$h_\beta P^1_{-1/2+i\tau}(\operatorname{ch}\alpha)\operatorname{ch}\tau\beta = \sum_{n=0}^\infty D^{(1)}_{2n+1}(\tau)\left(\frac{r}{a}\right)^{2n+1} P^1_{2n+1}(\cos\theta) \ (0 \leqslant r < a)$$

$$\left(\frac{a}{r}\right)^{2k+2} P^1_{2k+1}(\cos\theta) = h_\sigma \int_0^\infty C^{(1)}_{2k+1}(\tau) P^1_{-1/2+i\tau}(\operatorname{ch}\alpha)\operatorname{ch}\tau\sigma\, d\tau \quad (|\sigma| < \pi)$$

and satisfying the remaining conditions of 1), we arrive at the relationships

$$B_n = -\lambda^{2n+1}\int_0^\infty A(\tau) D^{(1)}_{2n+1}(\tau)\, d\tau + F_n \quad (n = 0, 1, 2, \ldots)$$

$$A(\tau) = -\frac{\operatorname{ch}\tau(\pi - \beta_0)}{\operatorname{ch}\tau\beta_0}\sum_{k=0}^\infty B_k \lambda^{2k+2} C^{(1)}_{2k+1}(\tau)$$

$$F_0 = -\varepsilon R, \quad F_n = 0 \ (n = 1, 2, \ldots) \quad (\lambda = R/a < 1,\ 0 < \beta_0 \leqslant \pi)$$

Eliminating the function $A(\tau)$ and setting

$$B_n = \frac{(-1)^{n+1}\varepsilon R}{\sqrt{(2n+1)(2n+2)}} b_n$$

we obtain an infinite system of linear algebraic equations to determine the unknown coefficients b_n

$$b_n = \sum_{k=0}^\infty c_{nk}(\lambda, \beta_0) b_k + f_n \quad (n = 0, 1, 2, \ldots) \tag{2.1}$$

$$c_{nk}(\lambda, \beta_0) = c_{kn}(\lambda, \beta_0) = 2\lambda^{2n+2k+3}[(2n+1)(2n+2)(2k+1)(2k+2)]^{1/2} \gamma_{nk}(\beta_0)$$

$$\gamma_{nk}(\beta_0) = \int_0^\infty \frac{(\tau^2 + 1/4)\operatorname{ch}\tau(\pi - \beta_0)}{\operatorname{ch}\tau\beta_0 \operatorname{ch}\tau\pi} F(-2n, 3/2 + i\tau; 3; 2) \times F(-2k, 3/2 \pm i\tau; 3; 2)\, d\tau$$

$$f_0 = \sqrt{2}, \quad f_n = 0 \quad (n = 1, 2, \ldots), \quad \lambda = R/a, \quad 0 < \beta_0 \leqslant \pi$$

For $0<\lambda<\delta=\min(1, \operatorname{tg}\beta_0/2)$ the inequality

$$\sum_{n,\,k=0}^{\infty} c_{nk}^2(\lambda, \beta_0)<\infty \tag{2.2}$$

holds.

Setting

$$\psi_m(\tau)=\sqrt{(\tau^2+1/4)\frac{\operatorname{ch}\tau(\pi-\beta_0)}{\operatorname{ch}\tau\beta_0\operatorname{ch}\tau\pi}}\,F(-2m, 3/2+i\tau; 3; 2)$$
$$(m=0, 1, 2, \ldots .)$$

and taking account of the inequality

$$\Big(\int_0^\infty \psi_n(\tau)\psi_k(\tau)\,d\tau\Big)^2\leqslant\int_0^\infty\psi_n^2(\tau)\,d\tau\int_0^\infty\psi_k^2(\tau)\,d\tau \quad (n, k=0, 1, 2, \ldots .)$$

we have the following estimates $(\gamma_{mm}(\beta_0)>0,\ c_{mm}(\lambda, \beta_0)>0)$

$$\gamma_{nk}^2(\beta_0)\leqslant\gamma_{nn}(\beta_0)\gamma_{kk}(\beta_0),\quad c_{nk}^2(\lambda, \beta_0)\leqslant c_{nn}(\lambda, \beta_0)c_{kk}(\lambda, \beta_0)$$

Therefore, to prove inequality (2.2), it is sufficient to see that

$$\sum_{k=0}^{\infty} c_{kk}(\lambda, \beta_0)<\infty \quad (0<\lambda<\delta) \tag{2.3}$$

By using the formulas /5/

$$\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n-p)}{n!\,\Gamma(-p)}s^nF(-n, b; -p; z)\,F(-n, \beta; -p; \zeta)=$$
$$(1+s)^{p+b+\beta}(1+s-sz)^{-b}(1+s-s\zeta)^{-\beta}F\Big[b, \beta; -p; -\frac{z\zeta s}{(1+s-sz)(1+s-s\zeta)}\Big] \tag{2.4}$$

(for $p=-3,\ z=\zeta=2,\ b=3/2+i\tau,\ \beta=3/2-i\tau,\ s=\pm\lambda^2,\ 0<\lambda<1$) we represent the left-hand side of inequality (2.3) as the sum of two components

$$\sum_{k=0}^{\infty} c_{kk}(\lambda, \beta_0)=g_1(\lambda, \beta_0)+g_2(\lambda, \beta_0)$$

$$g_1(\lambda, \beta_0)=\frac{2\lambda^3}{(1+\lambda^2)^3}\int_0^\infty\frac{(\tau^2+1/4)\operatorname{ch}\tau(\pi-\beta_0)}{\operatorname{ch}\tau\beta_0\operatorname{ch}\tau\pi}\times F\Big[3/2+i\tau, 3/2-i\tau; 3; \frac{4\lambda^2}{(1+\lambda^2)^2}\Big]d\tau$$

$$g_2(\lambda, \beta_0)=\frac{2\lambda^3}{(1-\lambda^2)^3}\int_0^\infty\frac{(\tau^2+1/4)\operatorname{ch}\tau(\pi-\beta_0)}{\operatorname{ch}\tau\beta_0\operatorname{ch}\tau\pi}\times F\Big[3/2+i\tau, 3/2-i\tau; 3; -\frac{4\lambda^2}{(1-\lambda^2)^2}\Big]d\tau$$

Using the equality /5/

$$\Gamma(\nu-m+1)\,m!\,P_\nu^{\ m}(x)=(-2)^{-m}\Gamma(\nu+m+1)(1-x^2)^{m/2}F(\nu+m+1, m-\nu; 1+m; 1/2-\frac{x}{2}) \tag{2.5}$$

(for $\nu=-1/2+i\tau,\ m=1,\ x=1-\frac{8\lambda^2}{(1+\lambda^2)^2},\ -1<x<1$) and the obvious estimate

$$0<F\Big[3/2+i\tau, 3/2-i\tau; 3; \frac{4\lambda^2}{(1+\lambda^2)^2}\Big]<F\Big[3/2+i\tau, 3/2-i\tau; 2; \frac{4\lambda^2}{(1+\lambda^2)^2}\Big]$$

we will have $(\theta_0=4\operatorname{arctg}\lambda,\ 0<\lambda<\delta,\ 0<\beta_0\leqslant\pi)$

$$0<g_1(\lambda, \beta_0)<\frac{\lambda^2}{1-\lambda^4}\int_0^\infty\frac{\operatorname{ch}\tau(\pi-\beta_0)}{\operatorname{ch}\tau\beta_0\operatorname{ch}\tau\pi}P^1_{-1/2+i\tau}(\cos\theta_0)\,d\tau<\infty$$

Setting $\gamma=1,\ \alpha=1/2+i\tau,\ \beta=1/2-i\tau,\ \zeta=-4\lambda^2(1-\lambda^2)^{-2}$ in the Gauss recursion formula /5–7/

$$\gamma(\gamma+1)[F(\alpha, \beta; \gamma; \zeta)-F(\alpha, \beta; \gamma+1; \zeta)]-\alpha\beta\zeta F(\alpha+1, \beta+1; \gamma+2; \zeta)=0 \tag{2.6}$$

and $\nu=-1/2+i\tau,\ z=1-2\zeta,\ m=0.1$ in the inequality /3/

$$P_\nu^{\ m}(z)=\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}\frac{(z^2-1)^{m/2}}{2^m m!}F\Big(m-\nu, m+\nu+1; m+1; \frac{1-z}{2}\Big)$$
$$(|\arg(z\pm1)|<\pi;\ m=0, 1, 2, \ldots .)$$

and also taking account of the functional relationship /3/

$$F(\alpha,\beta;\gamma;\zeta)=(1-\zeta)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma;\zeta)\ (|\arg(1-\zeta)|<\pi) \tag{2.7}$$

we represent the quantity $g_2(\lambda,\beta_0)$ in the form $(0<\beta_0\leqslant\pi)$

$$g_2(\lambda,\beta_0)=-\frac{1}{2}\operatorname{sh}\frac{\alpha_0}{2}\int_0^\infty\frac{\operatorname{ch}\tau(\pi-\beta_0)}{\operatorname{ch}\tau\beta_0\operatorname{ch}\tau\pi}\Big[P_{-1/2+i\tau}(\operatorname{ch}\alpha_0)+$$

$$\frac{\operatorname{cth}(\alpha_0/2)}{\tau^2+1/4}P^1_{-1/2+i\tau}(\operatorname{ch}\alpha_0)\Big]d\tau\quad(\alpha_0=4ar\operatorname{th}\lambda,\ 0<\lambda<1)$$

It follows from this equality and the asymptotic behaviour of the function $P^m_{-1/2+i\tau}(\operatorname{ch}\alpha_0)$ as $\tau\to\infty$ that $|g_2(\lambda,\beta_0)|<\infty$ $(0<\lambda<1)$.

Consequently, for $0<\lambda<\delta$ we have

$$\sum_{k=0}^\infty c_{kk}(\lambda,\beta_0)\leqslant g_1(\lambda,\beta_0)+|g_2(\lambda,\beta_0)|<\infty$$

It follows from inequality (2.2) and the fact that $\{f_n\}_0^\infty$ belongs to the Hilbert space of number sequences l_2 that for almost all values $\lambda\in(0,\delta)$ a solution of the infinite system (2.1) exists in l_2, is unique, and can be found by the method of reduction /8, 9/. The constraint $0<\lambda<\delta$ on the possible values of the parameter λ is related in a natural way to the formulation of the problem under consideration and means that the hemisphere $r=R$, $0\leqslant\theta\leqslant\pi/2$ lies entirely in the domain $0\leqslant\beta<\beta_0$.

The contact stresses $\tau_{r\varphi}=G(\partial u/\partial r-u/r)$ $(r=R,\ 0<\beta_0\leqslant\pi)$ as well as the relationship between the torque M applied to the stamp and the angle ε are determined directly in terms of the solution of the infinite system (2.1)

$$\tau_{r\varphi}|_{r=R}=G\varepsilon\sum_{n=0}^\infty\frac{(-1)^n(4n+3)}{\sqrt{(2n+1)(2n+2)}}b_nP^1_{2n+1}(\cos\theta),\ \varepsilon=-\frac{M\sqrt{2}}{4\pi R^3Gb_0}$$

We analyse problem 1) further for the limit case $\beta_0=\pi$, which corresponds to torsion of the half-space $z\geqslant0$ with the hemispherical depression $0\leqslant r\leqslant R$, $0\leqslant\theta\leqslant\pi/2$ under the boundary conditions

$$u|_{r=R}=\varepsilon R\sin\theta\quad\left(0\leqslant\theta\leqslant\frac{\pi}{2}\right),\quad\frac{\partial u}{\partial z}\Big|_{z=0}=0$$
$$(R<\rho<a),\quad u|_{z=0}=0\quad(\rho>a)$$

Using the estimate

$$|F(3/2+i\tau,\ -2k;3;2)|\leqslant\frac{\operatorname{ch}\pi\tau}{(2k+1)(4\tau^2+1)}$$

resulting from the integral representation

$$F(3/2+i\tau,-2k;3;2)=\frac{2\operatorname{ch}\pi\tau}{\pi(\tau^2+1/4)}\int_0^1t^{1/2-i\tau}(1-t)^{1/2+i\tau}(1-2t)^{2k}dt$$

and the inequality

$$\int_0^1\sqrt{t}\sqrt{1-t}(1-2t)^{2k}dt\leqslant\frac{\pi}{8(2k+1)}\quad(k=0,1,2,\ldots.)$$

it can be seen that for $n=0,1,2,\ldots$

$$\sum_{k=0}^\infty|c_{nk}(\lambda,\pi)|\leqslant1-\omega\quad\left(\omega=1-\lambda,\ 0<\lambda\leqslant\lambda_0=\frac{2}{\sqrt{4+\pi}}\right)$$

This inequality and the fact that the sequence $\{f_n\}_0^\infty$ belongs to the space l_1 assures single-valued solvability of the infinite system (2.1) in l_1 for $\beta_0=\pi$, $0<\lambda\leqslant\lambda_0$ and the applicability of the method of successive approximations /8, 9/.

Limiting ourselves to terms of order λ^9 inclusive in the solution of the infinite system (2.1), we will have

$$b_0=f_0[1+d_{00}\lambda^3+d_{00}^2\lambda^6+d_{00}^3\lambda^9+O(\lambda^{10})] \tag{2.8}$$
$$b_1=f_0d_{10}\lambda^5[1+d_{00}\lambda^3+O(\lambda^6)],\quad b_2=f_0d_{20}\lambda^7[1+O(\lambda^3)]$$

$$b_3 = f_0 d_{30}\lambda^9 [1 + O(\lambda^3)], \quad b_n = O(\lambda^{2n+3}) \quad (n = 4, 5, \ldots)$$

$$d_{00} = \frac{4}{3\pi}, \quad d_{10} = \frac{4\sqrt{6}}{15\pi}, \quad d_{20} = \frac{4\sqrt{15}}{35\pi}, \quad d_{30} = \frac{8\sqrt{7}}{63\pi}, \quad f_0 = \sqrt{2}$$

We express the shear stresses $\tau_{z\varphi} = G(\partial u/\partial z)(z = 0, \rho > a)$ directly in terms of the solution of the infinite system (2.1) and we investigate their behaviour as $\rho \to a$.

Taking account of the equality

$$\left.\frac{\partial P_{2k+1}^1(\cos\theta)}{\partial z}\right|_{z=0} = 0, \quad \frac{\partial \alpha}{\partial z} = 0, \quad \frac{\partial \beta}{\partial z} = \frac{\operatorname{ch}\alpha - 1}{a} \quad (z = 0, \ \rho > a)$$

we find

$$\tau_{z\varphi}|_{z=0,\ \rho>a} = -\frac{G}{a}(\operatorname{ch}\alpha - 1)^{3/2}\int_0^\infty \tau A(\tau)\operatorname{sh}\tau\pi\, P_{-1/2+i\tau}^1(\operatorname{ch}\alpha)\, d\tau$$

Now using the relationships

$$A(\tau) = -\frac{1}{\operatorname{ch}\tau\pi}\sum_{k=0}^\infty B_k \lambda^{2k+2} C_{2k+1}^{(1)}(\tau), \quad B_k = \frac{(-1)^{k+1}\varepsilon R}{\sqrt{(2k+1)(2k+2)}}\, b_k$$

$$\frac{\operatorname{sh}\tau\pi}{\operatorname{ch}\tau\pi} P_{-1/2+i\tau}^1(\operatorname{ch}\alpha) = -\frac{i}{\pi}[Q_{-1/2-i\tau}^1(\operatorname{ch}\alpha) - Q_{-1/2+i\tau}^1(\operatorname{ch}\alpha)]$$

and applying the residue theorem to evaluate the integral

$$R_k(\alpha) = -i\int_{-\infty}^\infty \frac{\tau F(3/2 + i\tau, -2k; 3; 2)}{\operatorname{ch}\tau\pi} Q_{-1/2-i\tau}^1(\operatorname{ch}\alpha)\, d\tau$$

we obtain the following formula to determine the shear stresses on the clamped part $z = 0, \rho > a$ of the boundary of the body under consideration

$$\tau_{z\varphi}|_{z=0,\ \rho>a} = \qquad (2.9)$$

$$-\frac{\sqrt{2}}{\pi} G\varepsilon(\operatorname{ch}\alpha - 1)^{3/2}\sum_{k=0}^\infty \sqrt{(2k+1)(2k+2)}\,\lambda^{2k+3} b_k R_k(\alpha)$$

$$R_k(\alpha) = \sum_{m=0}^\infty (-1)^m(2m+1) F(1-m, -2k; 3; 2) Q_m^1(\operatorname{ch}\alpha) \times$$

$$\left(\operatorname{ch}\alpha = \frac{\rho^2 + a^2}{\rho^2 - a^2}\right)$$

As $\rho \to a$ $(\alpha \to \infty)$ these stresses have an integrable singularity of the form $(\rho^2 - a^2)^{-1/2}$:

$$\tau_{z\varphi}|_{z=0,\ \rho>a} \sim \frac{2Ga\varepsilon}{\pi\sqrt{\rho^2 - a^2}}\sum_{k=0}^\infty \sqrt{\frac{2k+2}{2k+1}}\,\lambda^{2k+3} b_k \quad (\rho \to a)$$

Limiting ourselves to terms of the order of λ^9 inclusive in (2.9) and using the asymptotic solution (2.8) of the infinite system (2.1) for $\beta_0 = \pi$, we find

$$\tau_{z\varphi}|_{z=0,\ \rho>a} = \frac{4}{\pi} G\varepsilon\left(\frac{a}{\rho}\right)^3 \frac{a}{\sqrt{\rho^2 - a^2}}\left[\lambda^3 + \frac{4}{3\pi}\lambda^6 + \frac{16}{9\pi^2}\lambda^9 + O(\lambda^{12})\right]$$

The equality

$$\sum_{m=0}^\infty (-1)^m (2m+1) Q_m^1(\operatorname{ch}\alpha) = -\frac{\operatorname{sh}\alpha}{(\operatorname{ch}\alpha + 1)^2} \quad (\alpha > 0)$$

resulting from the Heine formula /5/

$$(\zeta - z)^{-1} = \sum_{m=0}^\infty (2m+1) P_m(z) Q_m(\zeta) \quad (|z + \sqrt{z^2 - 1}| < |\zeta + \sqrt{\zeta^2 - 1}|)$$

is used to obtain this result.

Let us note again the case $\beta_0 = \pi/2$ corresponding to the problem of the torsion of a hollow hemisphere $R \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant \pi/2$ by a circular stamp with hemispherical base $0 \leqslant r \leqslant R$,

$0 \leqslant \theta \leqslant \pi/2$. It can be seen that in this case

$$c_{nk}=0\ (n \neq k);\quad c_{nn}=\lambda^{4n+3}\quad (n,\ k=0,\ 1,\ 2,\ \ldots)$$

and, therefore, problem 1) has a closed solution for $\beta_0 = \pi/2$

$$b_0=\frac{a^3\sqrt{2}}{a^3-R^3},\quad b_n=0\ (n=1,\ 2,\ \ldots),\quad A(\tau)=2\sqrt{2}\,\frac{aR^3\varepsilon}{a^3-R^3}\,\frac{1}{\operatorname{ch}\tau\pi}$$
$$u=\frac{R^3\varepsilon\,(a^3-r^3)}{r^2\,(a^3-R^3)}\sin\theta\quad (R\leqslant r\leqslant a,\ R<a,\ 0\leqslant\theta\leqslant\pi/2)$$

We represent the general solution of problem 2) as the sum of two components

$$u=h_\sigma\int_0^\infty A(\tau)\,P^1_{-1/2+i\tau}(\operatorname{ch}\alpha)\operatorname{ch}\tau\sigma\,d\tau+\sum_{n=0}^\infty B_n\Big(\frac{r}{R}\Big)^{2n+1}P^1_{2n+1}(\cos\theta)$$

each of which satisfies the condition $(\partial u/\partial z)|_{z=0}=0$ $(a<\rho<R)$ identically.
Using the equalities

$$\beta=\pi-\sigma\,(0\leqslant\sigma\leqslant\pi),\quad \varepsilon\rho\,|_{\sigma=\sigma_0}=$$
$$-\,2^{3/2}a\varepsilon h_\sigma\int_0^\infty\frac{\operatorname{ch}\tau(\pi-\sigma_0)}{\operatorname{ch}\tau\sigma_0}\,P^1_{-1/2+i\tau}(\operatorname{ch}\alpha)\,d\tau$$
$$h_\sigma P^1_{-1/2+i\tau}(\operatorname{ch}\alpha)\operatorname{ch}\tau\sigma=\sum_{n=0}^\infty D^{(1)}_{2n+1}(\tau)\Big(\frac{a}{r}\Big)^{2n+2}P^1_{2n+1}(\cos\theta)\quad (r>a)$$
$$\Big(\frac{r}{a}\Big)^{2k+1}P^1_{2k+1}(\cos\theta)=h_\beta\int_0^\infty C^{(1)}_{2k+1}(\tau)\,P^1_{-1/2+i\tau}(\operatorname{ch}\alpha)\operatorname{ch}\tau\beta\,d\tau\quad (|\beta|<\pi)$$

satisfying the remaining conditions of 2), and setting

$$B_n=\frac{(-1)^n a\varepsilon}{\sqrt{(2n+1)(2n+2)}}\,b_n$$

we obtain, after some reduction, an infinite system of linear algebraic equations $(\lambda=R/a<1,\ 0<\sigma_0\leqslant\pi)$

$$b_n=\sum_{k=0}^\infty c_{nk}(\lambda,\ \sigma_0)\,b_k+f_n(\lambda,\ \sigma_0)\quad (n=0,\ 1,\ 2,\ \ldots)\tag{2.10}$$
$$c_{nk}(\lambda,\ \sigma_0)=c_{kn}(\lambda,\ \sigma_0)=2\lambda^{2n+2k+3}\,[(2n+1)(2n+2)\times$$
$$(2k+1)(2k+2)]^{1/2}\gamma_{nk}(\sigma_0)$$
$$\gamma_{nk}(\sigma_0)=\int_0^\infty\frac{(\tau^2+{}^1/_4)\operatorname{ch}\tau(\pi-\sigma_0)}{\operatorname{ch}\tau\sigma_0\operatorname{ch}\tau\pi}\,F(-2n,\ {}^3/_2+i\tau;\ 3;\ 2)\times$$
$$F(-2k,\ {}^3/_2\pm i\tau;\ 3;\ 2)\,d\tau$$
$$f_n(\lambda,\ \sigma_0)=4\sqrt{(2n+1)(2n+2)}\,\lambda^{2n+2}\gamma_{n0}(\sigma_0)$$

Because the matrices $\|c_{nk}(\lambda,\ \sigma_0)\|$ and $\|c_{nk}(\lambda,\ \beta_0)\|$ agree, apart from the notation of the parameters therein, and the sequence $\{f_n(\lambda,\ \sigma_0)\}$ is an element of the spaces l_1 and l_2, the infinite system (2.10) possesses the same properties as the infinite system (2.1). It is merely necessary to replace the parameters R, a, β_0 by a, R, σ_0 respectively, in the formulation of the properties for the latter. We note that in the problems considered there is a complete analogy even for the investigation of the shear stresses.

We will represent the general solution of problem 3) as the sum of two components

$$u=h_\sigma\sum_{n=0}^\infty H_n P^1_{n-1/2}(\operatorname{ch}\alpha)\cos n\sigma+\sum_{k=0}^\infty G_k\Big(\frac{r}{R}\Big)^{2k+1}P^1_{2k+1}(\cos\theta).$$

each of which satisfies the condition $(\partial u/\partial z)_{z=0}=0$ $(0\leqslant\rho<a\operatorname{th}(\alpha_0/2),\ a\operatorname{cth}(\alpha_0/2)<\rho<R)$ identically.

Using the equalities

$$\rho=-\frac{2a\sqrt{2}}{\pi}\,h_\sigma\Big[Q^1_{-1/2}(\operatorname{ch}\alpha)+2\sum_{n=1}^\infty Q^1_{n-1/2}(\operatorname{ch}\alpha)\cos n\sigma\Big]$$
$$h_\sigma P^1_{n-1/2}(\operatorname{ch}\alpha)\cos n\sigma=\sum_{k=0}^\infty b^{(1)}_{2k+1,\,n}\Big(\frac{a}{r}\Big)^{2k+2}P^1_{2k+1}(\cos\theta)\,(r>a)$$

$$\left(\frac{r}{a}\right)^{2k+1} P^1_{2k+1}(\cos\theta) = h_\sigma \sum_{n=-\infty}^{\infty} (-1)^n a^{(1)}_{2k+1,\,n} Q^1_{n-1/2}(\operatorname{ch}\alpha)\cos n\sigma$$
$$(\alpha > 0)$$
$$Q^1_{-n-1/2}(\operatorname{ch}\alpha) = Q^1_{n-1/2}(\operatorname{ch}\alpha), \quad a^{(1)}_{2k+1,\,-n} = a^{(1)}_{2k+1,\,n}$$

satisfying the remaining conditions of 3), and setting

$$H_n = \frac{2a\varepsilon\sqrt{2}}{\pi}\operatorname{sgn}\left(n^2 - \frac{1}{4}\right)\left[-\left(n^2-\frac{1}{4}\right)^{-1}\frac{Q^1_{n-1/2}(\operatorname{ch}\alpha_0)}{P^1_{n-1/2}(\operatorname{ch}\alpha_0)}\right]^{1/2} h_n$$

we arrive at an infinite system of linear algebraic equations

$$h_n = \sum_{m=0}^{n} p_{nm}(\lambda,\ \alpha_0) h_m + q_n(\alpha_0) \quad (n = 0,\ 1,\ 2,\ \ldots) \tag{2.11}$$
$$p_{0m}(\lambda,\ \alpha_0) = \frac{2}{\pi}\sqrt{\psi_0(\alpha_0)\psi_m(\alpha_0)}\,\omega_{0m}(\lambda) \quad (m = 0,\ 1,\ 2,\ \ldots)$$
$$p_{nm}(\lambda,\ \alpha_0) = \frac{4}{\pi}\sqrt{\psi_n(\alpha_0)\psi_m(\alpha_0)}\,\omega_{nm}(\lambda)$$
$$(m = 0,\ 1,\ 2,\ \ldots,\ n = 1,\ 2,\ \ldots)$$
$$\omega_{nm}(\lambda) = \sum_{k=0}^{\infty} \lambda^{4k+3}(2k+1)(2k+2) F(3/2 - n,\ -2k;\ 3;\ 2) \times$$
$$F(3/2 - m,\ -2k;\ 3;\ 2)$$
$$q_0(\alpha_0) = \sqrt{\psi_0(\alpha_0)}, \quad q_n(\alpha_0) = 2\sqrt{\psi_n(\alpha_0)} \quad (n = 1,\ 2,\ \ldots)$$
$$\psi_k(\alpha_0) = -\left(k^2 - \frac{1}{4}\right)\frac{Q^1_{k-1/2}(\operatorname{ch}\alpha_0)}{P^1_{k-1/2}(\operatorname{ch}\alpha_0)} > 0, \quad \lambda = \frac{a}{R} < 1$$

Applying (2.4), and then the functional relationships /3/

$$F(\alpha,\ \beta;\ \gamma;\ z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\ \gamma-\beta;\ \gamma;\ z)$$
$$F(\alpha,\ \beta;\ \gamma;\ z) = (1-z)^{-\alpha} F\left(\alpha,\ \gamma-\beta;\ \gamma;\ \frac{z}{z-1}\right)$$
$$(|\arg(1-z)| < \pi)$$

we will have an explicit expression for the quantities $\omega_{nm}(\lambda)$ in terms of the hypergeometric function

$$\omega_{nm}(\lambda) = \frac{\lambda^3}{(1+\lambda^2)^3}\left(\frac{1-\lambda^2}{1+\lambda^2}\right)^{n+m} F\left[3/2 + n,\ 3/2 + m;\ 3;\ \frac{4\lambda^2}{(1+\lambda^2)^2}\right] +$$
$$\frac{\lambda^3}{(1+\lambda^2)^3}\left(\frac{1+\lambda^2}{1-\lambda^2}\right)^{n-m} F\left[3/2 - n,\ 3/2 + m;\ 3;\ \frac{4\lambda^2}{(1+\lambda^2)^2}\right]$$

Therefore, the matrix elements of the infinite system (2.11) are also expressed in explicit form.

For $0 < \lambda < \operatorname{th}(\alpha_0/2)(\alpha_0 > 0)$ the matrix $\|p_{nm}(\lambda,\ \alpha_0)\|$ satisfies the condition

$$\sum_{n,\ m=0}^{\infty} p^2_{nm}(\lambda,\ \alpha_0) < \infty \tag{2.12}$$

Actually, by setting

$$\xi_k(x) = \lambda^{2k+3/2}\sqrt{(2k+1)(2k+2)}\, F(3/2 - x,\ -2k;\ 3;\ 2)$$

in the Cauchy inequality

$$\left(\sum_{k=0}^{\infty} \xi_k(n)\xi_k(m)\right)^2 \leqslant \sum_{k=0}^{\infty} \xi_k^2(n) \sum_{k=0}^{\infty} \xi_k^2(m)$$

we obtain the following estimates $(p_{kk}(\lambda,\ \alpha_0) > 0)$:

$$p^2_{nm}(\lambda,\alpha_0) \leqslant p_{nn}(\lambda,\alpha_0)\, p_{mm}(\lambda,\alpha_0) \quad (n, m = 0, 1, 2, \ldots)$$

Therefore, to prove the inequality (2.12) it is sufficient to show that for $0 < \lambda < \operatorname{th}(\alpha_0/2)$

$$\sum_{n=1}^{\infty} p_{nn}(\lambda,\alpha_0) < \infty \tag{2.13}$$

By using (2.5)-(2.7) we have $(\theta_0 = 4\operatorname{arctg}\lambda)$

$$\sum_{n=1}^{\infty} p_{nn}(\lambda,\alpha_0) = \frac{4}{\pi}\frac{\lambda^3}{(1+\lambda^2)^3}\sum_{n=1}^{\infty}\psi_n(\alpha_0)\left(\frac{1-\lambda^2}{1+\lambda^2}\right)^{2n} \times$$
$$F\left[3/2 + n,\ 3/2 + n;\ 3;\ \frac{4\lambda^2}{(1+\lambda^2)^2}\right] + \frac{4}{\pi}\sin\frac{\theta_0}{2}\sum_{n=1}^{\infty}\frac{\psi_n(\alpha_0)}{1-4n^2} \times$$
$$\left[P_{n-1/2}(\cos\theta_0) + 4\operatorname{ctg}\frac{\theta_0}{2}\frac{P^1_{n-1/2}(\cos\theta_0)}{4n^2-1}\right]$$

It hence follows that the last series converges for all $\lambda \in (0, 1)$. Now using the inequality $(n \geqslant 1,\ 0 < \lambda < 1)$

$$F\left[{}^3/_2 + n, {}^3/_2 + n; 3; \frac{4\lambda^2}{(1+\lambda^2)^2}\right] < 2F\left[{}^1/_2 + n, {}^1/_2 + n; 1; \frac{4\lambda^2}{(1+\lambda^2)^2}\right]$$

resulting from the definition of the hypergeometric series /3, 5/, the representation of the Legendre function $P_\nu^\mu(z)$ in terms of the hypergeometric function /3, 5/

$$P_\nu^\mu(z) = \frac{2^{-\nu}}{\Gamma(1-\mu)}(z+1)^{{}^1/_2\mu+\nu}(z-1)^{-{}^1/_2\mu} F\left(-\nu, -\nu-\mu; 1-\mu; \frac{z-1}{z+1}\right)$$

and the relationship $P_{-\nu-1}(z) = P_\nu(z)$, we obtain the following estimate

$$\left(\frac{1-\lambda^2}{1+\lambda^2}\right)^{2n} F\left[{}^3/_2 + n, {}^3/_2 + n; 3; \frac{4\lambda^2}{(1+\lambda^2)^2}\right] < 2\,\mathrm{ch}\,\alpha_1 P_{n-{}^1/_2}(\mathrm{ch}\,2\alpha_1)\ (\alpha_1 = 2\,\mathrm{arth}\,\lambda)$$

Taking account of the asymptotic behaviour of the toroidal functions $P_{n-{}^1/_2}^m(\mathrm{ch}\,\alpha)$, $Q_{n-{}^1/_2}^m(\mathrm{ch}\,\alpha)$ this estimate indeed proves condition (2.13) for $0 < \lambda < \mathrm{th}\,(\alpha_0/2)$ $(0 < \alpha_1 < \alpha_0)$.

It follows from inequality (2.12) and the fact that $\{q_n(\alpha_0)\}_0^\infty$ belongs to the space l_2 that for almost all values of $\lambda \in (0, \mathrm{th}\,(\alpha_0/2))$ the solution of the infinite system (2.11) exists in l_2 is unique, and can be found by the method of reduction.

We note that the problem of the torsion of a half-space $z \geqslant 0$ with hemispherical $(0 \leqslant r \leqslant R,\ 0 \leqslant \theta \leqslant \pi/2)$ and torodial $(0 < \alpha_0 \leqslant \alpha < \infty,\ 0 \leqslant \sigma \leqslant \pi)$ depressions by stamps coupled to the surfaces of these depressions also reduces to an infinite system of linear algebraic equations with the matrices $\|p_{nm}(\lambda, \alpha_0)\|$ $(\lambda = R/a)$.

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